Math308, Quiz 8, 03/28/14

First Name: .... **Last Name: ....**

Table 1: Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s} \quad s > 0$	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}} \quad s > 0$
$e^{-\alpha t}$	$\frac{1}{s+\alpha} \quad s > -\alpha$	$e^{-\alpha t} t^n$	$\frac{n!}{(s+\alpha)^{n+1}} \quad s > -\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2} \quad s > 0$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2} \quad s > 0$
$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2} \quad s > \alpha$	$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2} \quad s > 0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2} \quad s > \lvert\omega\rvert$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
$u_\alpha(t)$	$\frac{e^{-\alpha s}}{s} \quad s > 0$	$\delta(t-\alpha)$	$e^{-\alpha s} \quad s > -\infty$

Theorem. Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose that there exist constants K, a and M such that $|f(t)| \le Ke^{at}, |f'(t)| \le Ke^{at}, \ldots, |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}[f^{(n)}(t)]$ exists for $s > a$ and given by

- $\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$.
- if $F(s) = \mathcal{L}[f(t)]$ for $s > a$, then $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} \mathcal{L}[f(t)] = e^{-cs} F(s)$ for $s > a, c > 0$.
- if $f(t) = \mathcal{L}^{-1}[F(s)]$, then $u_c(t) f(t-c) = \mathcal{L}^{-1}[e^{-cs} F(s)]$.

Grade: ....

Show all work!

Problem 1. 100%. Use the Laplace transform to solve the following initial value problem:

$$\begin{aligned} &y'' - 2y' + 2y = 0 \\ &y(0) = 0, \quad y'(0) = 1. \end{aligned} \tag{1}$$

Solutions

By taking the Laplace transform of the equation we obtain

$$\mathcal{L}[y''] - 2\mathcal{L}[y'] + 2\mathcal{L}[y] = 0. \tag{2}$$

We use the above theorem to express $\mathcal{L}[y'']$ and $\mathcal{L}[y']$ in terms of $\mathcal{L}[y]$:

$$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - 2\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = 0, \tag{3}$$

or we can simplify it as

$$\left(s^2 - 2s + 2\right)\mathcal{L}[y] - (s-2)y(0) - y'(0) = 0.$$

And we now apply the initial conditions and find $\mathcal{L}[y]$:

$$L[y] = \frac{1}{s^2 - 2s + 2},$$

or

$$L[y] = \frac{1}{(s-1)^2 + 1}.$$

Finally, the inverse Laplace transform of the right hand side is

$$y = e^t \sin t.$$