# Math308

Deadline: April 28, 2014

## Programming Assignment: Initial-Value Problem for Ordinary Differential Equations

Consider the initial-value problem

$$y'(t) = f(t, y), \; y(t_0) = y_0, \text{ from } t = t_0 \text{ to } t = t_f. \tag{1}$$

## 1 Develop and Test your code

### 1.1 Euler's Method

The mfile `euler.m` implements Euler's method to solve the first order initial value problem (1). Verify the code works correctly by comparing the results with **Table 8.1.1** on page 445 of the book (Boyce & DiPrima) for problem (1) with

$$f(t, y) = 1 - t + 4y, \quad y_0 = 1, \text{ from } t = 0 \text{ to } t = 2.$$

Your code should work for arbitrary choice of the step-size $h$.

### 1.2 Improved Euler's Method

Now, implement the improved Euler's methods described in Chapter 8.2, page 454. Use your code for the improved Euler's method to solve the problem (1) with the following data:

$$f(t, y) = 1 - t + 4y, \quad y_0 = 1, \text{ from } t = 0 \text{ to } t = 2.$$

Verify your code works by reproducing the results in **Table 8.2.1** of the book.

### 1.3 Runge-Kutta Method (RK4)

Now, implement the Runge-Kutta method described in Chapter 8.3, page 459. Use your code for the Runge-Kutta method to solve the problem (1) with the following data:

$$f(t, y) = 1 - t + 4y, \quad y_0 = 1, \text{ from } t = 0 \text{ to } t = 2.$$

Verify your code works by reproducing the results in **Table 8.3.1** of the book.

### 1.4 Table 1:error

### 1.5 Plot 1: Convergence Rates for Euler Methods

First, find the exact solution to the problem

$$\frac{dy}{dt} = 1 - t + 4y, \quad y_0 = 1. \tag{2}$$

Include the exact solution in your report.

Now that you've developed working code, you will compare the error in each of the methods as described below. Denote the computed solutions by $y_E$ for the Euler's code, $y_{ME}$ for the modified Euler's code, $y_{RK}$ for the Runge-Kutta code, and $y_{exact}(t)$ the exact solution.

Assume the interval $[0, 1]$ is divided by $N$ equal sub-intervals and denote the length of the sub-intervals by $h$. Then, $h = t_i - t_{i-1} = \frac{1}{N}$ for any $i = 1, 2, \ldots, N$. Now, use your program to compute the approximate solution of the IVP (2) for $N$ = 5, 10, 20, 40, 80, 160, 320, 640, 1280, 2560 using both Euler's and modified Euler's methods. Plot $h$ versus $h^\alpha$ together with $h$ versus the error $|y_{exact}(t_N) - y_{approx}(t_N)|$ in *loglog*-plot in Matlab. (Notice you should have $t_N = 1$). Use $\alpha = 1$ for Euler and $\alpha = 2$ for modified Euler. Include both of these plots in your report, and discuss your findings.

### 1.6 Plot 2: Convergence Rates for Runge-Kutta Method

Using the Runge-Kutta method, use your program to compute the approximate solution of the IVP (2) from $t = 0$ to $t = 1$ for $N$ = 5, 10, 20, 40, 80, 160, 320, 640, 1280, 2560. Plot $h$ versus $h^\alpha$ and $h$ versus the error $|y_{exact}(t_N) - y_{RK}(t_N)|$, in *loglog*-plot in Matlab for a different values of $\alpha$ and determine the value of $\alpha$ for which you best match the slope of the error. Include in your report the plot of the $h$ vs. the error $|y_{exact}(t_N) - y_{RK}(t_N)|$ and $h$ versus $h^\alpha$ for this value of $\alpha$. Report the value of $\alpha$ you found and discuss your findings. *Hint: type the matlab command* `help plot` *for some useful plotting commands.*

## 2 Thermodynamics Problem

This problem is adapted from *Scientific Computing with Matlab*, Quarteroni and Saleri, 2003. Consider a body having internal temperature $T$ which is set in an environment with constant temperature $T_e$. Assume that its mass $m$ is concentrated in a single point. Then the heat transfer between the body and the external environment can be described by the Stefan-Boltzmann law

$$v(t) = \epsilon\gamma S(T^4(t) - T_e^4) \tag{3}$$

where $t$ is the time variable, $\epsilon$ the Boltzmann constant for which you may use $\epsilon = 5.6 * 10^{-8} J/m^2K^4s$, where $J$ stands for Joule, $K$ for Kelvin, $m$ for meter, and $s$ for second. $\gamma$ is the emissivity constant of the body, $S$ the area of its surface and $v$ is the rate of heat transfer. The rate of variation $E(t) = mCT(t)$, where $C$ denotes the specific heat of the material constituting the body, equals in absolute value to the rate $v$. Consequently, setting $T(0) = T_0$, the computation of $T(t)$ requires the solution of the ordinary differential equation

$$\frac{dT}{dt} = -\frac{v(t)}{mC}, \tag{4}$$

with $v(t)$ given by (3). Consider the case where the body in question is a cube with sides equal to 1 $m$ and and mass equal to 1 $Kg$. Assume $T_0 = 180K,\ T_e = 200K,\ \gamma = 0.5$ and $C = 100J/(Kg/K)$.

### 2.1 Analytical Results

Notice that (4) is a separable equation, and use this to determine a solution for $T(t)$ in *implicit* form. Feel free to use a table of integrals or other means to evaluate any difficult integrals you may encounter. Include the implicit solution you found in your report, and describe how you found it.

Take a moment to contemplate the implications if you were to have to solve *explicitly* for $T(t)$. Now that we have sufficiently motivated numerical methods for approximating the solutions to differential equations, proceed to solve the problem with your code as follows:

### 2.2 Numerical Results

Use your code for the Euler method, the Improved Euler method and the Runge-Kutta method to solve equation (4) with the given parameters. Compare the results obtained by using $h = 20$ and $h = 10$, for $t$ ranging from 0 to 200 seconds. Include a table of these results in your report for each value of $h$, including the computed values of $T$ at 20, 100 and 200 seconds using each of the methods and discuss your findings.

***Some useful advice for your report***. Please, be clear when you write your report, motivate your answer, do not submit a plain code with no explanation, put axis, title and legend to all figures, explain what is plotted, try to minimize your computational data in your presented table, *e.g.,* no table should be more than one page.

Your report should include at least three plots, two tables, the exact solution to each problem (one in *explicit* form and one in *implicit* form) and the source code. Discuss your findings when you present data.